A Quasi-Center for Planar Convex Sets

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Abstract

The center of a convex set is the point, if such a point exists, that bisects every chord through that point; this definition is very restrictive, and the center exists only for highly symmetrical convex sets. This paper introduces the quasi-center as a generalization of the center, for compact convex planar sets. Suppose we have a point in such a set, and a chord through that point. Then the point divides the chord into two segments, and one can find the ratio of the shorter segment to the entire chord; call this ratio the cut-ratio. This cut-ratio is between 0 and 0.5, and the nearer it is to 0.5, the better that point does at bisecting that chord. For a particular point, we can take the minimum cut-ratio, over all the chords through that point, as an overall estimate of how closely that point bisects the chords through it. The minimum cut-ratio is a function whose domain is the points in a compact, convex set. The point with the highest minimum cut-ratio will be called the quasi-center. This paper shows that a compact, convex planar set (though not necessarily a set in higher dimensions) has a unique quasi-center, and presents a geometrically intuitive construction to find that quasi-center. Several examples show that the quasi-center is different from the center of mass, although the two are usually very close. A notable feature of the quasi-center is that it does not require any Euclidean concepts like length and angle: a vector space and its operations are sufficient for its definition and construction.

1 Introduction

A convex set $\mathcal{C}$ is a subset of $\mathbb{R}^n$ such that, if $\mathcal{C}$ contains two points p_1 and p_2, then $\mathcal{C}$ also contains the line segment joining p_1 and p_2. Convex sets are "solid," in the sense that they contain no holes or inlets, and boast many regularities, such as well-defined areas, perimeters, and integer dimensions. Furthermore, they can often be specified conveniently and concisely, as linear combinations (whose coefficients are non-negative and sum to 1) of a set of points, or as the intersection of a set of half-spaces.

In many simple, regular cases, such as circles or parallelograms, a convex set has a natural center. Even for an irregular convex set, however, geometric intuition tends to identify a center, or at least a small region that is centrally located. Various definitions have formalized the notion of center. Technically, *the* center of a convex set $\mathcal{C}$ (see p. 54 of Ref. 1, and Ex. 13.11 of Ref. 2) is the point p that bisects every chord through p, where a chord is the intersection of $\mathcal{C}$ with a line through the interior of $\mathcal{C}$. While a natural definition, such a center rarely exists. In this paper, this definition of center will be referred to as the *bisection center*.

This paper proposes a new definition, the *quasi-center*, that provides a reasonable notion of "center" for any compact convex planar set $\mathcal{C}$. The quasi-center is the result of a visually appealing, dynamic geometric construction, that involves sliding shrunken copies of $\mathcal{C}$ along the boundary of $\mathcal{C}$. The quasi-center was developed to make up for the shortcomings of the bisection center. While a point of $\mathcal{C}$ that bisects every chord it lies on usually does not exist, it seemed that there should exist a point that comes as close as possible to bisecting every chord. This paper proves not only that such a point exists, but that it is unique, and this point was named the quasi-center.

More formally, suppose that p is a point of a compact convex planar set $\mathcal{C}$, and that ch is a chord through p. Then p cuts ch into two segments. The ratio of the shorter segment to the total chord will be called the *cut-ratio*, and denoted α. Of course, there are many chords through p, and many cut-ratios. The minimum cut-ratio, over all the chords through p, will be denoted $M(p)$. $M(p)$ can be thought of as the most uneven division of a chord that p accomplishes. $M(p)$ is always positive and bounded above by 0.5. If $M(p) = 0.5$, then p is a bisection center, because it cuts each chord through it into two equal sections. If $M(p)$ is near zero, then p cuts at least one chord very unevenly, so p is not a good candidate for the center. We will define the quasi-center of $\mathcal{C}$ to be the point $p \in \mathcal{C}$ that maximizes $M(p)$, and the value of M at the quasi-center will be denoted μ.

A geometric method will be given that finds the quasi-center. Denote a level set of the function $M(p)$ by M_α, for a given value α; formally, $M_\alpha = \{p \in \mathcal{C} | M(p) \geq \alpha\}$. It will be shown that M_α is a closed convex set, that can be constructed geometrically, as shown in Figure 1. In that figure, $\mathcal{C}$ is the large triangle. The left side of the figure shows some grey triangles, which are translations of $\alpha\mathcal{C}$, the homothety of $\mathcal{C}$ by the factor α. The homothety $\alpha\mathcal{C}$ is slid around the perimeter of $\mathcal{C}$, sweeping out the grey region shown on the right of the figure. The remaining region, that is not coloured grey, is the set M_α. M_α can be thought of as indicating a central region of $\mathcal{C}$, that is at some distance from the boundary of $\mathcal{C}$. As α increases, the grey region on the right becomes larger, and the set M_α becomes smaller. Eventually, α will reach some value μ, which is probably less than 0.5, at which M_μ is a single point—that point is the quasi-center.

The definitions and constructions in this paper have been carefully formulated to avoid metric concepts such as length or angle. This restriction arose from practical considerations. While analyzing some colour constancy algorithms, examples arose of convex sets in a chromaticity diagram. The chromaticity diagram was a vector space of dimension two, but there was no natural idea of the distance between two chromaticities, nor the angles between vectors terminating at different chromaticities. Any operations on the convex sets could therefore involve only vector space properties.

While convex sets are often considered in a Euclidean setting, where concepts like perpendicularity or shortest distance are defined, convexity can also be considered in a pure vector space setting, with no Euclidean structure. Many seemingly Euclidean definitions, such as the bisection center, in fact do not need Euclidean notions at all. For example, the ratio of the "length" of a segment of a chord to the "length" the chord itself can be recast as a linear relationship between two parallel vectors; the coefficient in that relationship can provide the ratio, without assigning lengths to vectors. To insure wide applicability, this pa-

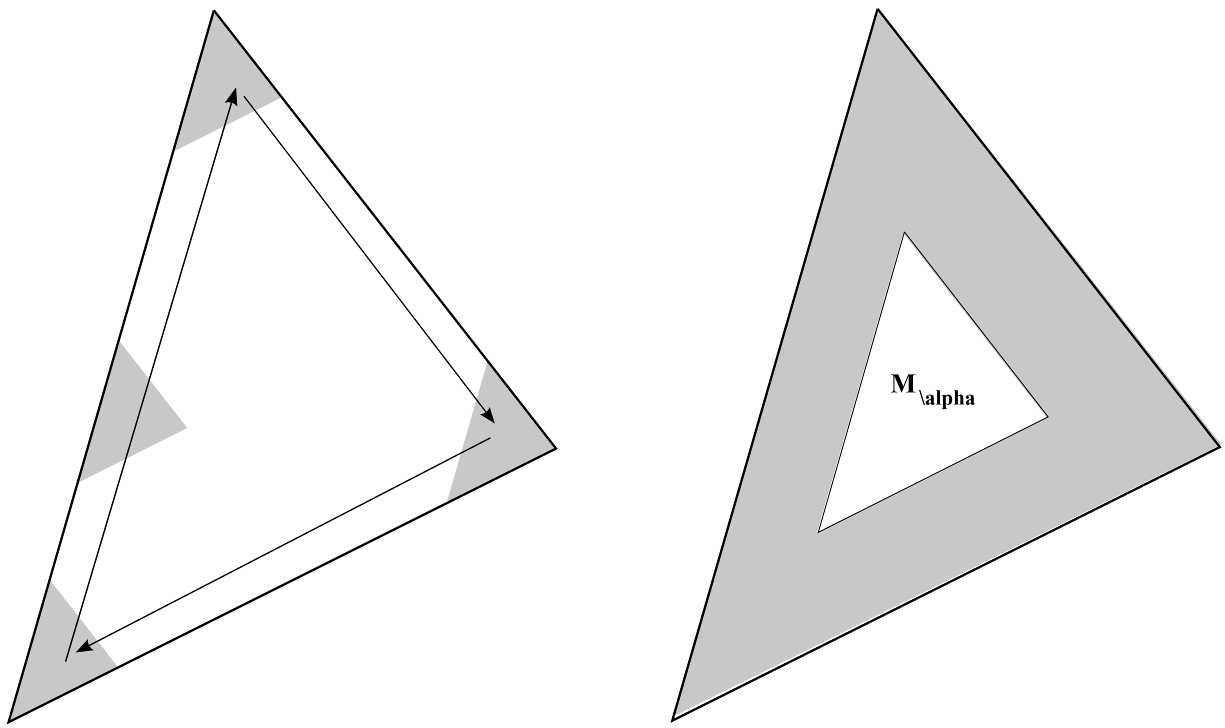

Figure 1: A Dynamic Geometric Construction for M_α

per works only with vector spaces, using only vector space properties; no metric structures are imposed.

This paper is organized as follows. First, convexity is discussed in a vector space setting. Next, some current definitions of centers for convex sets are described, and the definition of the quasi-center is introduced. Then, a dynamic geometric construction for the quasi-center is developed; this construction is used to prove that the quasi-center for a planar set exists and is unique, and also provides a practical algorithm for calculating it. Higher-dimensional extensions are discussed, and it is seen that a quasi-center is not necessarily unique for non-planar sets. Finally, the quasi-center is compared with other definitions. Standard results about convexity are used freely, usually without specific attribution; such results can be found in References 1 and 2.

2 Definitions

2.1 Convex Sets in a Vector Space

Convex sets were originally defined in the context of Euclidean geometry, and many results in convexity only apply in a Euclidean space. To define a convex set, however, a Euclidean space is not needed: a vector space by itself has sufficient structure. This paper will restrict itself to the vector space setting, and will not assume that any metric notions such as length or angle are present. As a consequence, the results and algorithm will apply very broadly. This section will outline the vector space properties that are needed for the development of the quasi-center.

The vector space approach was motivated by an application of convex sets to colour constancy algorithms.[3] The *chromaticity* of a colour stimulus encompasses that colour's hue (red, orange, green, etc.) and saturation (whether the colour is a vivid version of a hue, or a dull, greyed-down version), but not its lightness (how light or dark that colour is). The set of chromaticities can be represented in a *chromaticity diagram*, which is a two-dimensional vector space. In this vector space, however, there is no distinguished basis, and no natural idea of the distance between two chromaticities, nor the angles between chromaticity vectors. The *grey-world hypothesis* is an heuristic that asserts that the set of chromaticities in a camera image is on average the chromaticity of the illumination under which that image was taken. The set of all possible chromaticities must be a convex set, and estimating the illuminant requires finding that set's average or center, in some sense. Since the chromaticity diagram has no Euclidean structure, the center of this convex set must be found using only vector space properties.

The main requirement to define convexity is that a unique line segment can be drawn between any pair of points, and a vector space allows such segments to be constructed. Suppose we have a real vector space $\mathbb{R}^n$, and two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$. Then define the line segment between $\mathbf{v}_1$ and $\mathbf{v}_2$ by

$$S(\mathbf{v}_1, \mathbf{v}_2) = \left\{ \sum_{i=1}^{2} \beta_i \mathbf{v}_i \middle| 0 \leq \beta_i \leq 1 \ \forall i \text{ and } \sum_{i=1}^{2} \beta_i = 1 \right\}. \tag{1}$$

Geometrically, this set gives a parameterized path from the interval $[0, 1]$ into V, as can be seen by rewriting Equation (1) as

$$S(\mathbf{v}_1, \mathbf{v}_2) = \{\beta_1 \mathbf{v}_1 + (1 - \beta_1)\mathbf{v}_2 | 0 \leq \beta_1 \leq 1\}. \tag{2}$$

The path starts at $\mathbf{v}_2$, when $\beta_1 = 0$, and ends at $\mathbf{v}_1$, when $\beta_1 = 1$. The vectors defined by Equation (1) are called *convex combinations* of $\mathbf{v}_1$ and $\mathbf{v}_2$. If an inner product provided the vector space with a Euclidean structure, then this path would correspond to the Euclidean straight line segment between the points.

This algebraic line segment is sufficient to make the following definition: a subset $\mathcal{C}$ of $\mathbb{R}^n$ is *convex* if and only if, for every pair of points $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{C}$, the straight line segment given by Equation (1) is also in $\mathcal{C}$. For this paper, we will further require $\mathcal{C}$ to be compact, in the standard topology of $\mathbb{R}^n$. Compactness implies that $\mathcal{C}$ is closed, and so it contains its boundary. More importantly, it implies that $\mathcal{C}$ is bounded, which is necessary for a center to exist at all. Another restriction is that $\mathcal{C}$ should have non-empty interior; this restriction could easily be lifted, however, by working in the smallest affine subspace that contains $\mathcal{C}$.

One consequence of convexity is that a straight line that intersects the interior of a compact convex set $\mathcal{C}$ also intersects $\mathcal{C}$ in exactly two boundary points. A *chord* of $\mathcal{C}$ is defined to be the intersection of $\mathcal{C}$ with such a straight line. If b_1 and b_2 are the two boundary points, then the corresponding chord ch is just the line segment $S(b_1, b_2)$.

Even if a vector space V has no metric structure imposed on it, operations such as vector addition and scalar multiplication are still possible. In addition, relations of linear dependence and independence, between vectors or sets of vectors, can be determined. Parallel translation is always defined, and one can always write a vector as a unique scalar multiple of a parallel vector. These properties allow us to divide a line segment in any ratio desired. Furthermore, if a point is on a line segment, we can determine the ratio in which that point cuts the segment; this ability will be used when we investigate how near a would-be center comes to bisecting the chords through that center. We can also use the ratio property to construct homotheties of a set, based at a given point.

2.2 Current Definitions of the Center of a Convex Set

Currently, there are several common definitions of the center of a compact convex set $\mathcal{C}$, some of which require a Euclidean metric. The Chebyshev center, for instance, is the center of the incircle or circumcircle of $\mathcal{C}$, but Euclidean notions are required to define incircles and circumcircles. A related center can be defined in terms of Löwner-John ellipsoids,[4] but again, Euclidean notions are required to evaluate how well such an ellipsoid fits $\mathcal{C}$. Only two definitions, the *bisection center* and the *centroid* (or perhaps more correctly, the *center of mass*) are metric-free. This section presents those definitions. Later sections will compare them with the proposed quasi-center.

2.2.1 The Bisection Center

What this paper calls the bisection center of $\mathcal{C}$ is typically called just *the* center. It is the point $c \in \mathcal{C}$, if such a c exists, that bisects any chord through c. If c exists, then it is unique. (A simple proof: draw the chord ch through two possible centers, c_1 and c_2. Both c_1 and c_2 bisect ch, so they must both be located at its unique midpoint, and are therefore identical.) In the cases where it exists, the bisection center is a simple, natural definition. As a general definition, however, the bisection center is largely inadequate, because it rarely exists, unless $\mathcal{C}$ exhibits rotational symmetry. Even then, many highly symmetrical set, such as regular polygons with an odd number of sides, which one would expect to have a bisection center, turn out not to. For example, a regular pentagon has no bisection center, although a regular hexagon does. The bisection center, however, does have the advantage of being metric-free. Its bisection test only requires determining the ratio in which a point divides a chord, which we have seen is defined in a general vector space, without a concept of length. Though its application is limited, it seems desirable that any other definition of center should reduce to the bisection center whenever possible, as we will see that the quasi-center does.

2.2.2 The Centroid, or Center of Mass

Another common definition, that is often taken as the center of $\mathcal{C}$, especially in physical applications, is the *centroid*. The centroid can account for cases when $\mathcal{C}$ has a varying distribution, say of mass or electrical charge. When the distribution is uniform, the term *center of mass* is sometimes used instead. For simplicity, this paper will both use the term centroid, and assume that any distribution over $\mathcal{C}$ is uniform.

Suppose that $\mathcal{C}$ is a compact convex subset of $\mathbb{R}^n$, and that $\mathbb{R}^n$ has coordinates $x_i, x_2, \ldots, x_n$. Then Cartan's exterior algebra allows us to define a *volume form*, which is a multilinear, alternating function of n vectors. Typically the volume form is written $dx_1 dx_2 \ldots dx_n$. A volume form allows integrals to be evaluated over *chains*, such as compact convex sets. This formalism allows us to define the coordinates x_i^c of the *centroid* of $\mathcal{C}$ by

$$x_i^c = \frac{\int_{\mathcal{C}} x_i dx_1 dx_2 \dots dx_n}{\int_{\mathcal{C}} dx_1 dx_2 \dots dx_n}. \tag{3}$$

(To be thorough, we must also prove that this definition is independent of the basis. Suppose there is another basis, given by a change-of-basis matrix T. Indicate the new basis by the superscript 2. Then the two volume forms $dx_1 dx_2 \dots dx_n$ and $dx_1^2 dx_2^2 \dots dx_n^2$ are related by the multiplicative factor $\det(T)$. Then the constant $\det(T)$ will divide out of the numerator and denominator in Equation (3), leaving $x^{c2} = Tx^c$, which is the same vector as the original centroid, written in the new coordinate system.)

Like the bisection center, the centroid can be defined without using any metric structure. Another, more abstract, way to define x^c is to assert that $f(x^c)$ is the average value of $f(\mathcal{C})$, for any linear functional f on $\mathbb{R}^n$. A physical interpretation of this abstract definition is that the planar shape $\mathcal{C}$ will balance perfectly when supported at the centroid x^c. This physical interpretation is also a natural motivation for thinking of the centroid as a center of $\mathcal{C}$.

If $\mathcal{C}$ has a bisection center c, then either the physical interpretation or the evaluation of Equation (3) makes it clear that the centroid occurs at c. Of course, the centroid exists for an arbitrary compact convex set, while the bisection center does not. We will see later that the quasi-center agrees with the centroid in some important cases, but that there are other important cases in which the two differ.

2.3 The Quasi-Center of a Compact Convex Planar Set

The previous sections have described a compact, convex set $\mathcal{C}$ as a subset of a finite-dimensional vector space V, and have introduced the centroid and bisection center as two notions for a central point of $\mathcal{C}$. Neither definition requires any Euclidean structure. This section defines the quasi-center, which is another possible center for $\mathcal{C}$, which also requires no metric considerations. The quasi-center is similar to the bisection center, but relaxes the conditions: rather than requiring that a point p *exactly* bisects every chord through p, the quasi-center is the point p that comes *as close as possible* to bisecting every chord through p. Following sections will show that the quasi-center is unique for planar sets (but not necessarily unique for non-planar sets), and present a dynamic geometric algorithm for finding it.

To define the quasi-center formally, let p be a point in the interior of $\mathcal{C}$, and let ch be a chord through p, that intersects the boundary of $\mathcal{C}$ in two points, b_1 and b_2. Then p will divide $S(b_1, b_2)$ into two vectors, $v(b_1, p)$ and $v(p, b_2)$. While no absolute length can be assigned to either vector, length comparisons can still be made, because the vectors are parallel. Relabel the boundary points if necessary so that $v(b_1, p)$ is shorter than (or the same length as) $v(p, b_2)$. Then define the *cut-ratio* $\alpha(p, \mathrm{ch})$ for p and ch by

$$v(b_1, p) = \alpha(p, \mathrm{ch}) v(b_1, b_2). \tag{4}$$

The cut-ratio is the "length" of the shorter segment of the chord, divided by the "length" of the total chord. For any particular p and ch, the cut-ratio is between 0 and 0.5. It equals 0.5 only when p bisects ch.

Equation (4) describes only one chord through p. To see if p is a good candidate for the center of $\mathcal{C}$, we should consider its behavior on every chord through p. Therefore define a minimum cut-ratio $M(p)$ by

$$M(p) = \min_{\mathrm{ch}} \alpha(p, \mathrm{ch}), \tag{5}$$

where ch is any chord through p. $M(p)$ can never be greater than 0.5, and equals 0.5 only when p is the bisection center in the restricted sense defined earlier. $M(p)$ gives the most uneven division of a chord that p accomplishes. It is easy to see that M is a continuous function on $\mathcal{C}$.

We will define the quasi-center of $\mathcal{C}$ as the point in $\mathcal{C}$ at which M takes on a maximum value μ. The existence of such a maximum follows because M is a continuous function on a compact set. To speak of *the* quasi-center instead of *a* quasi-center requires a proof of uniqueness, which following sections will supply for planar sets.

3 A Geometrical Construction for the Quasi-Center

The previous section defined the (or at least *a*) quasi-center for a compact, convex set $\mathcal{C}$, but gave no way to find it. This section presents a visually appealing geometric approach to locating the quasi-center.

3.1 The Sliding Homothety Construction

A simple two-dimensional example, in which $\mathcal{C}$ is the triangle shown in Figure 2, will explain the geometric algorithm. To begin with, let us find P_α, the set of all the points $p \in \mathcal{C}$ whose minimum cut-ratio is at least α :

$$P_\alpha = \{p \in \mathcal{C} | M(p) \geq \alpha\}. \tag{6}$$

For purposes of illustration, we will initially set $\alpha = 0.2$, and then let α be arbitrary. Suppose there is a particular chord ch, as shown in Figure 2, intersecting the boundary of $\mathcal{C}$ at b_1 and b_2. The point p divides ch into two segments, such that the segment from b_1 to p is 0.2 times the "length" of the entire chord. Then it is clear that no point in the (open) segment between b_1 and p can be in $P_{0.2}$, because the minimum cut-ratio μ of such a point is less than 0.2 on the chord ch.

This argument applies to every chord, but for now let us apply it solely to the chords that originate at b_1, as shown in Figure 3. $P_{0.2}$ cannot contain any point in the first twenty percent of any chord through b_1. Figure 3 shows in grey the segments that consist of the first twenty percent of each chord through b_1. Denote this grey set by $\mathcal{C}(b_1, 0.2)$. As a whole, the segments in $\mathcal{C}(b_1, 0.2)$ form a copy of $\mathcal{C}$, and the size of that copy is twenty percent of the size of the original $\mathcal{C}$. In technical language, $\mathcal{C}(b_1, 0.2)$ is a homothety of $\mathcal{C}$ through the point b_1. The boundaries of $\mathcal{C}$ and $\mathcal{C}(b_1, 0.2)$ overlap around b_1. An important fact about $\mathcal{C}(b_1, 0.2)$ is that $P_{0.2}$ must lie completely outside it, because any point in $\mathcal{C}(b_1, 0.2)$ cuts some chord in a ratio less than 0.2.

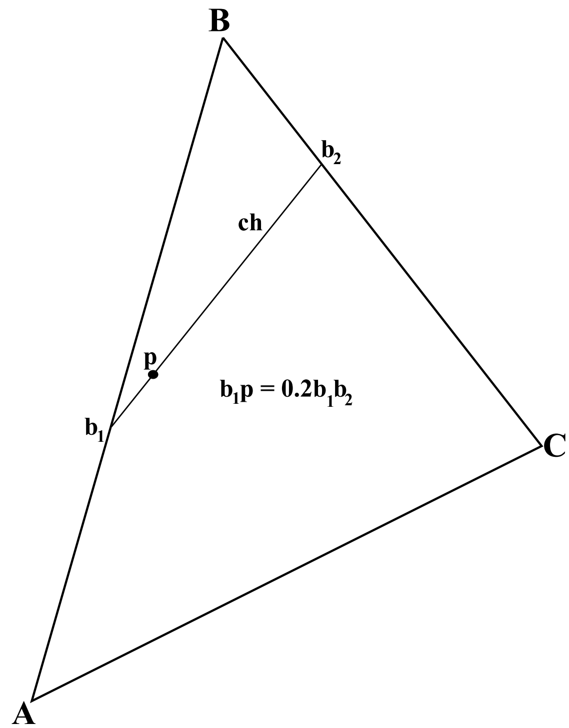

Figure 2: A Chord and Dividing Point with Cut-Ratio 0.2

This homothety construction can be applied not just at b_1, but at any point b on the boundary of $\mathcal{C}$. The result in every case will be a shrunken version of $\mathcal{C}$, at the point b. Even when b is a vertex, $\mathcal{C}(b, 0.2)$ will itself contain a duplicate of that vertex, which fits perfectly inside the vertex at b. The boundary at b might be a curve instead of a straight line. In that case, $\mathcal{C}(b_1, 0.2)$ will be tangent to $\mathcal{C}$ at b, and the curvature of the boundary of $\mathcal{C}(b_1, 0.2)$ at b will be greater than the curvature of $\mathcal{C}$ at b (assuming the curvature is defined). In all cases, the homothetic copy will be contained completely within $\mathcal{C}$.

We have seen that $P_{0.2}$ lies completely outside $\mathcal{C}(b_1, 0.2)$, and similar arguments show that $P_{0.2}$ lies completely outside $\mathcal{C}(b, 0.2)$, for any boundary point b. A dynamic interpretation of the sets $\mathcal{C}(b, 0.2)$ allows us to combine all these exclusions. Start with $\mathcal{C}(A, 0.2)$, where A is a vertex of the triangle, as shown on the left side of Figure 4. Slide $\mathcal{C}(A, 0.2)$ along the side AB of the triangle, towards the vertex B. At any point of this motion, the translated $\mathcal{C}(A, 0.2)$ will lie directly on top of $\mathcal{C}(b, 0.2)$, for some b on AB. Similarly, given any b on AB, a translated copy of $\mathcal{C}(A, 0.2)$ will at some point overlie the homethety at b. Thus this sliding motion covers every point in every homothety along AB, and no other points. Continue this translation around the triangle, similarly sweeping out areas along sides BC and CA, ending up at A where we began. Denote by $S_{0.2}$ the total region that has been swept out; the right side of the figure shows $S_{0.2}$, shaded grey.

We can now show that $P_{0.2}$ is the set given by $\mathcal{C}/S_{0.2}$. Since every homothetic copy consists of points that cut at least one chord in a ratio less than 0.2,, it must be that no point in $S_{0.2}$ is in $P_{0.2}$. Conversely, suppose that a point p is in $P_{0.2}$. Then p cannot cut any chord in a ratio less than 0.2. By its construction, however, $S_{0.2}$ encompasses *every* point in $\mathcal{C}$, that cuts *any* chord in a ratio less than 0.2. Therefore p cannot be in $S_{0.2}$. Since every point is

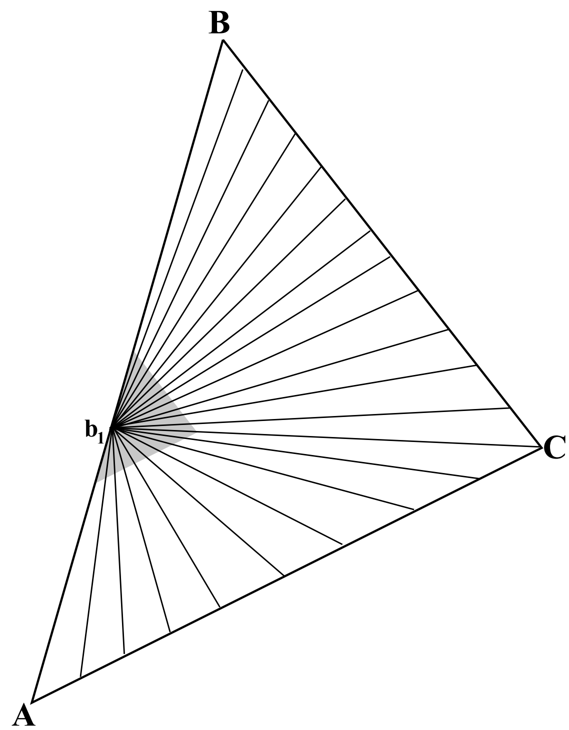

Figure 3: The First 20 Percent of All Chords Originating at b_1

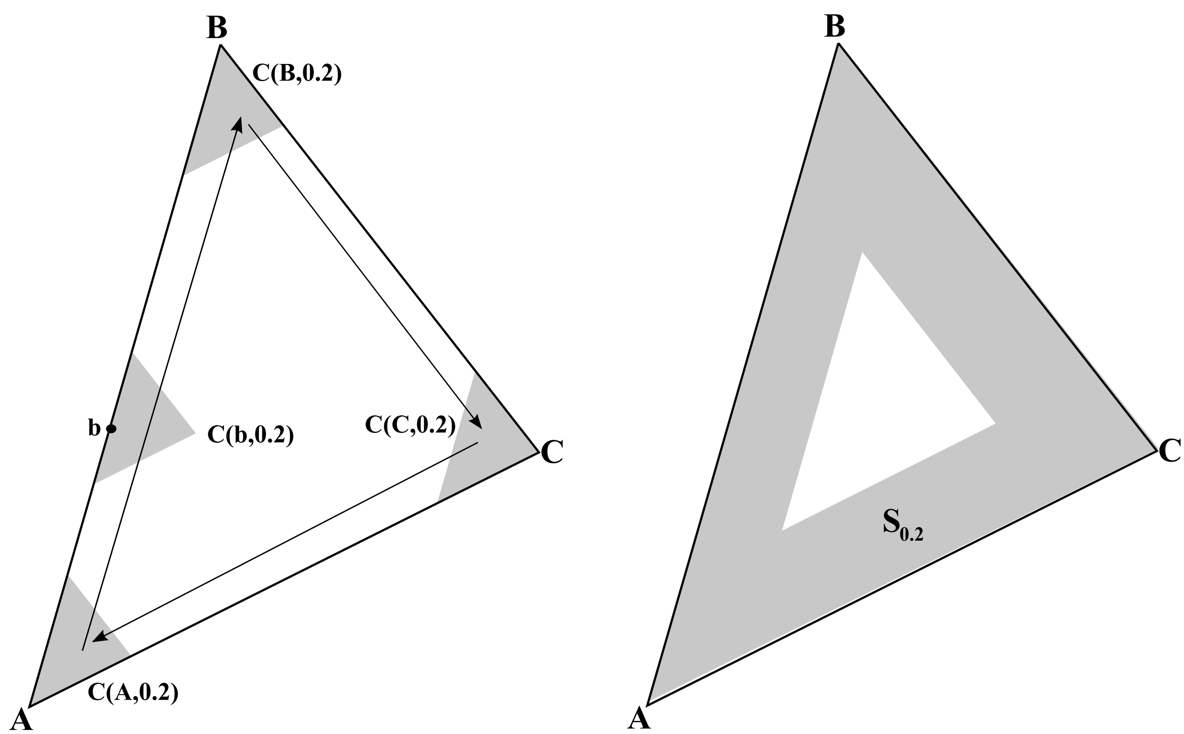

Figure 4: The Sliding Homothety Construction

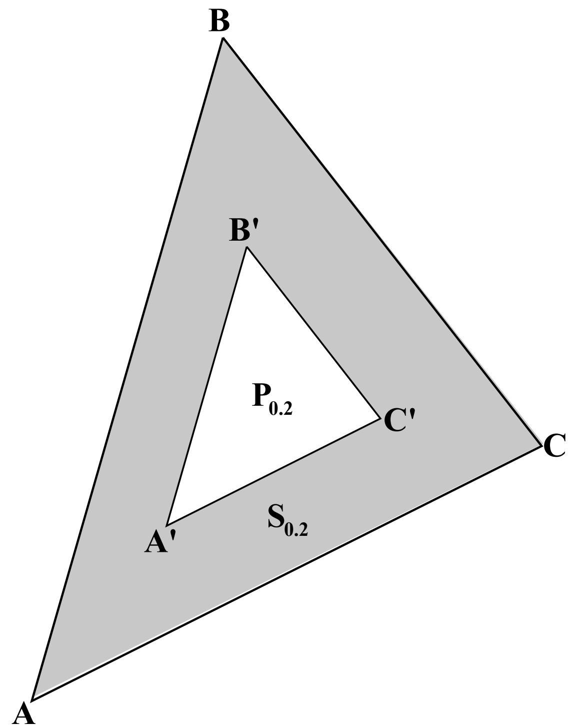

Figure 5: The Set $P_{0.2}$

in either $P_{0.2}$ or $S_{0.2}$, but not both, it must be that $P_{0.2}$ consists exactly of that part of $\mathcal{C}$ that is outside $S_{0.2}$. Figure 5 indicates $P_{0.2}$ for the triangle in Figure 2. Note that we have defined $P_{0.2}$ so that it is a closed set, that contains its boundary

This sliding homothety construction allows us to draw further conclusions about $P_{0.2}$, in particular that it is a closed convex set. Since $S_{0.2}$ was constructed by sliding homothetic copies of $\mathcal{C}$ along the edges of the triangle $\mathcal{C}$, it follows that each inner boundary of $S_{0.2}$ is a translation of an edge of the triangle. For example, the inner boundary $A'B'$ is a parallel translation of the edge AB of the triangle. Since any point that is in $\mathcal{C}$ but outside $S_{0.2}$ is in $P_{0.2}$, therefore an inner boundary of $S_{0.2}$ is an outer boundary of $P_{0.2}$, or is at least outside $P_{0.2}$. Suppose $A'B'$ were extended to infinity in either direction. This extension would create a closed half-space that contains $P_{0.2}$; the half-space is closed by definition, because it is assumed to contain the extended $A'B'$. By creating such a closed half-space from $B'C'$ and $C'A'$ as well, $P_{0.2}$ could be written as the intersection of three closed half-spaces. Such an intersection of half-spaces is always a convex set; furthermore, the intersection of a collection of closed sets is again closed. It thus follows that $P_{0.2}$ in this example is a closed convex set, as was to be shown.

While the demonstration of the previous paragraph has implicitly used the fact that $\mathcal{C}$ is a polygon, the extension to non-polygons is clear: simply approximate any boundary segment that is not a straight line by a polygonal set of straight lines. A set of closed half-spaces can then be found, whose intersection is a closed convex set that approximates $P_{0.2}$. By using progressively finer polygonal approximations to the boundary of $\mathcal{C}$, one can find a sequence of closed convex sets, which converge to $P_{0.2}$, which is therefore itself closed and convex.

Although the example given is two-dimensional, extensions to higher dimensions are also clear. In three dimensions, one would use polyhedra, while in an arbitrary dimension, one would use polytopes. Homothetic copies of a convex set at a boundary point are defined in any dimension, as are cut-ratios, and there are no problems translating those copies over the entire boundary. The foregoing constructions therefore apply in an arbitrary number of dimensions.

The sliding homothety construction in Figure 5 is reminiscent of the spherical neighborhood construction with radius $-\epsilon$ (see Sect. 14.9 of Ref. 1), in which the center of a ball of radius ϵ is slid over the boundary of a convex set. Any area that the ball sweeps out is then removed. While at a casual glance Figure 5 seems to be implementing such a construction, there is in fact a significant difference. Apart from the fact that we have no metric structure with which to define balls of radius ϵ, the value of ϵ itself would be different for different edges of the triangle, even if there were a metric structure. In the example, the inner boundary corresponding to AB is traced out by the image of C under the homotheties. The "distance" between AB and the corresponding inner boundary is twenty percent of the "distance" from AB to C. Similarly, the "distance" between AC and its corresponding inner boundary is twenty percent of the "distance" from AC to B. If this triangle were located in standard Euclidean space, then the distance from AC to B would be an altitude, and the radius ϵ for each side would be twenty percent of the altitude of the remaining vertex. In general, of course, these ϵs would be different for each side, although constant over any one side.

3.2 Converging on the Quasi-Center

The previous section showed how to construct the set P_α of points of $\mathcal{C}$ whose minimum cut-ratio μ is α or greater. Recall that the quasi-center of $\mathcal{C}$ is the point of $\mathcal{C}$ at which μ attains its maximum value. This section will show that, as α increases, the sets P_α form a nested sequence of closed convex sets, that converge on a single point, which is the quasi-center.

From the definition of μ, it is clear that $P_{\alpha 1} \supseteq P_{\alpha 2}$ whenever $\alpha_1 \leq \alpha_2$. The geometry of the construction makes this fact even clearer: as α increases, the region S_α swept out by the sliding homotheties extends further inwards, away from the boundary of $\mathcal{C}$, so the region left for P_α decreases. When indexed by α, the sets P_α thus form a nested sequence, in which each P_α is contained in all the P_αs before it.

Since α is a cut-ratio, it takes on a maximum value of 0.5, as does μ. If α were greater than 0.5, then the sliding homethety construction would still be defined, but the set P_α would be empty, because the inner boundaries of S_α would have crossed one another, so the intersection of the resulting half-planes would be empty. As α increases from 0, in fact, P_α will become empty when $\alpha = \mu$, and stay empty thereafter.

When the bisection center is defined, such as occurs for a circle of radius R, $\mu = 0.5$, and $P_{0.5}$ is the one-point set consisting of the center. Geometrically, $S_{0.5}$ in this case is the region swept out when a circle of radius $R/2$ (a homothety of the original circle by a factor of 0.5) is slid along the circumference of the circle, maintaining tangency at the point of contact (see Figure 6). The only remaining point, that is not swept out, is the center of the circle.

While P_α is always empty for $\alpha > 0.5$, it is also possible for P_α to be empty when $\alpha < 0.5$. As an example, Figure 7 shows Figure 5, when α is 0.4 instead of 0.2. The homothetic copies, like $\mathcal{C}(A, 0.4)$, are twice the size of the homothetic copies, like $\mathcal{C}(A, 0.2)$, in Figure

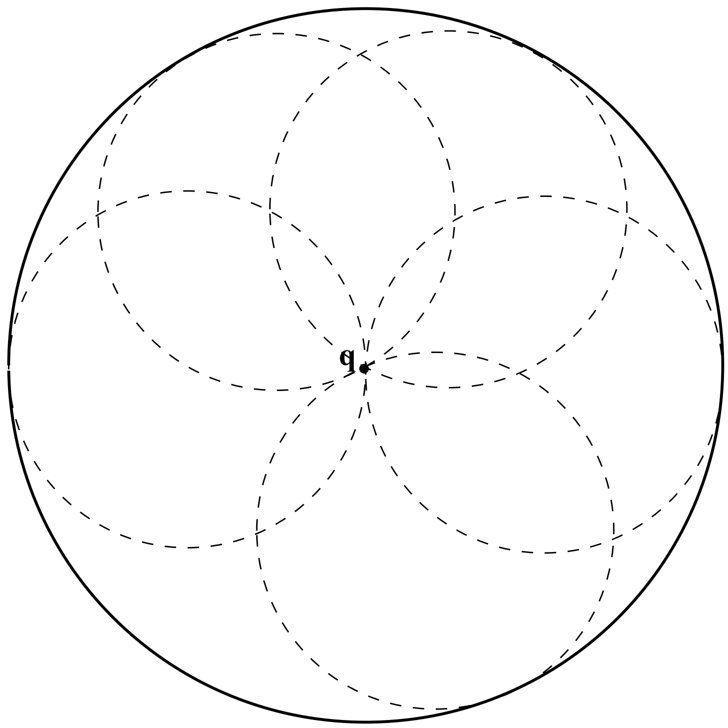

Figure 6: The Sliding Homothety Construction for a Circle when $\alpha = 0.5$

5. The striped area in the center of the triangle will actually be covered three times, as the homothetic copy slides along the three sides. The inner boundary $A'B'$ is twice as far from AB in Figure 7 as it is in Figure 5, as are the inner boundaries corresponding to the other two sides. The closed half-space corresponding to $A'B'$ is to the right of $A'B'$, and does not intersect the other two half-spaces, so $P_{0.4}$ is empty. Suppose p is any point in the triangle; then an implication is that there will always be at least one chord ch through p, such that p cuts ch in a ratio less than 0.4.

Since $\mu(p)$ is the minimum cut-ratio α over all chords through p, and since all ps whose minimum cut-ratio is greater than α are in the set P_α, therefore $\mu(p)$ takes on a maximum when P_α is as small as possible, which occurs at the smallest α for which P_α is empty. The maximum value of μ is then that smallest α. These statements can be seen more clearly geometrically. As α increases from 0, S_α will expand until it covers all of $\mathcal{C}$. The infimum of all the αs which makes $S_\alpha = \mathcal{C}$ is the maximum value of the function μ. That infimum occurs at the first α at which P_α has empty interior, and we will show that this P_α is a single point, the quasi-center.

Algorithmically, the critical value of α can be determined by bisection; conceptually, it is more natural to increase α steadily from 0 until P_α collapses to a single point. Figure 8 shows the results for the triangle in the original example. Since $\alpha = 0.2$ is too small, and $\alpha = 0.4$ is too big, we will try an intermediate value of $\alpha = 1/3$. The figure shows the 1/3 homothetic copies of the triangle in each of the vertices, and indicates with dotted lines the inner boundaries of $S_{1/3}$ corresponding to the triangle's three edges. Since the set $S_{1/3}$ is open, it can be seen that every point of the triangle is covered, except the point marked q. $P_{1/3}$ is therefore the single point q. If α were less than 1/3, even by a slight amount, then

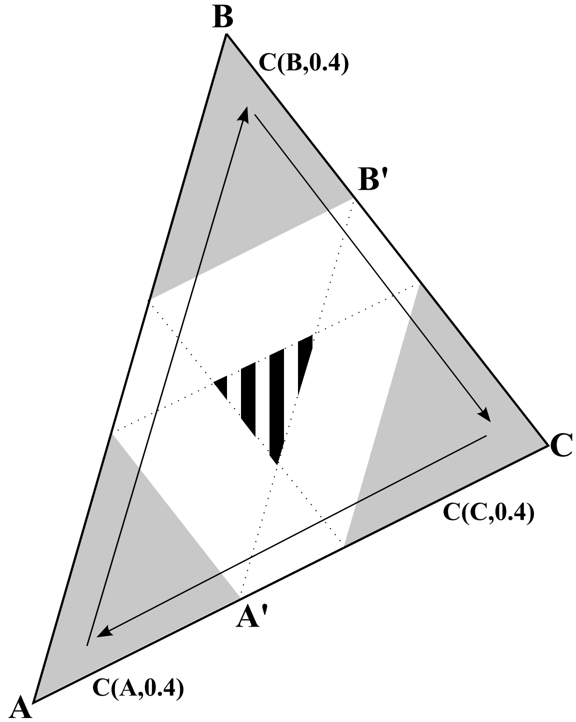

Figure 7: The Sliding Homothety Construction when $\alpha = 0.4$

P_α would include some small open neighborhood around q.

The conclusion is that the maximum μ for this triangle is $1/3$: the closest we can come to finding a point that bisects every chord through it is the point q, which cuts at least one chord in the ratio $1/3$. It can be seen that q does bisect some of its chords, but by no means all of them. The cut-ratios for q's chords vary from a maximum of 0.5 to a minimum of $1/3$, and we cannot find any point with a higher minimum cut-ratio. Since q satisfies this condition, we conclude that q is the quasi-center of the triangle.

3.3 Proof that the Quasi-Center is Unique

This section will prove that the compact convex planar set $\mathcal{C}$ has a unique quasi-center. Uniqueness is equivalent to asserting that the set P_m is a single point. Assume, by way of contradiction, that P_m is not a single point. Then, since P_m is a convex set, it must contain a line segment L. Then L is also contained in every P_α, for $\alpha < \mu$. As α increases, and S_α gradually covers P_α, L can only remain inside P_α as long as least one of the bounding edges of P_α (such as the edges $A'B'$, $B'C'$, and $C'A'$ in the figure) is parallel to L; otherwise, each edge, and therefore the boundary as a whole, will intersect L at some point, so that L is not contained in some P_α. Each bounding edge of P_α, as shown earlier, is parallel to some edge of $\mathcal{C}$, so the boundary of $\mathcal{C}$ must contain a line segment L_1 that is parallel to L, and whose copy in P_α converges to L.

When $\alpha = m$ and the copy of L_1 reaches L, the construction insures that there is at least one chord ch, with an initial point b_1 on L_1, and an intermediate point p on L. The terminal point b_2 of ch is on the boundary of $\mathcal{C}$, and can be found by extending $S(b_1, p)$ by a factor of

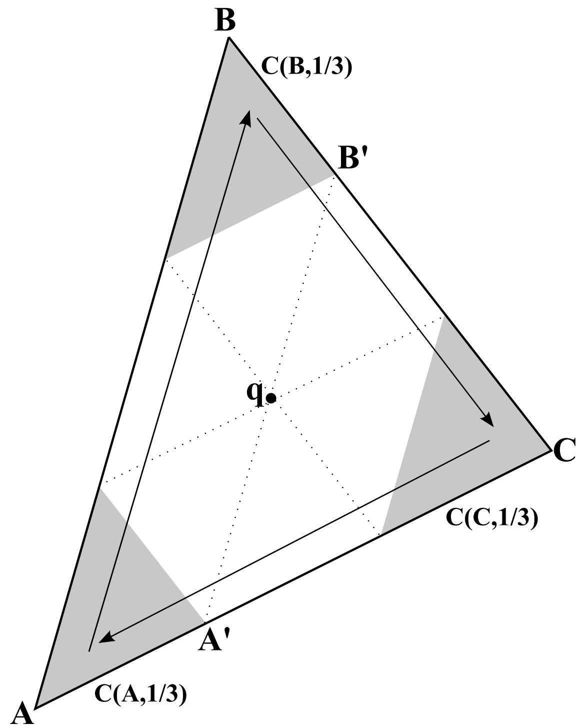

Figure 8: The Sliding Homothety Construction when $\alpha = 1/3$

$1/m$. In fact, since L is a line rather than a single point, there must be a continuous set of such chords, all translations of ch in the direction given by L. There is therefore a continuous set of b_2s, in the direction given by L, tracing out a line segment L_2 on the boundary of $\mathcal{C}$. We can loosely think of L_2 as being on the "opposite side" of $\mathcal{C}$ from L_1.

L_1 and L_2 are reversible, in the sense that we can think of those same chords as originating on L_2 and terminating on L_1. L_1 and L_2 each produce a translated copy that is on the boundary of P_m. The translated copy of L_1 will be m of the way towards L_2, and similarly the translated copy of L_2 will be m of the way towards L_1. These two copies will be separated as long as $m < \mu$. In order for P_μ to have no interior, the copy of L_2 in P_μ must reach L at exactly the same time as the copy of L_1 does. By the reversibility, however, the two copies can only reach each other when $m = 1/2$. Since $1/2$ is the maximum value that μ can obtain, then μ for $\mathcal{C}$must be $1/2$, implying that $\mathcal{C}$ has a bisection center. Since a bisection center is a unique point when it exists, the "line" L must in fact be a single point q, which is the unique quasi-center.

This proof does not extend to higher dimensions. The reason is that the boundary segments L_1 and L_2 are unique in two dimensions, but might not be unique in more dimensions. A simple counter-example, which shows that the quasi-center is not unique in three dimensions, is a triangular prism. In this case, the previous triangular constructions can be adapted to show that the set $P_{1/3}$ is the line segment consisting of the middle third of the prism's axis. Any point on this line segment is a quasi-center that maximizes the minimum cut-ratio, so the quasi-centers are not unique. Many lines on the boundary of the prism could serve as L_1 or L_2, and there is no reason to expect L_1, L_2, and $P_{1/3}$ to fall in the same plane, as they must in the two-dimensional case.

4 Comparison with Previous Definitions

This section compares the newly-defined quasi-center with two previous definitions, the bisection center and the centroid. It will be shown that the quasi-center generalizes the bisection center, which only exists for a limited number of convex sets. More importantly, it will be shown that the quasi-center is distinct from the centroid, although the two agree in some non-trivial cases. The differences between the two are slight in all the cases examined, but some examples will show that they are not identical. Both definitions are independent of metric structures. The centroid has the advantages, which the quasi-center lacks, of applying also to non-convex sets, and being uniquely defined for a set of arbitrary dimension.

If the bisection center exists for a certain convex set $\mathcal{C}$, then it can easily be seen that the bisection center is also the quasi-center. This is a limiting case, because μ reaches its maximum possible value of 0.5. Likewise, if μ is 0.5, then $P_{0.5}$ is a single point whose cut-ratio for every containing chord is 0.5, so $P_{0.5}$ is in fact the bisection center. For most convex sets, of course, the bisection center does not exist, while the quasi-center exists for any compact convex planar set. Thus the quasi-center generalizes the bisection center to a much wider variety of convex sets.

The relationship between the quasi-center and the centroid is more complicated. Figure 8 shows that the two points are sometimes the same, in a non-trivial case. Besides being the quasi-center, the point q in that figure is also the centroid. This result follows from analysis of the similar triangles in the figure. It can be seen that the line Bq, if extended, would bisect the edge AC, so q is a median of the triangle. Similar arguments show that q is also on the other two medians. Since the three medians of a triangle intersect at the centroid, q must be the centroid. In this triangle, then, the quasi-center and centroid agree.

In fact, the metric-independence of the quasi-center and the centroid show that they agree for *any* triangle. If a general planar shape is moved so that its centroid is at the origin, then the average value of any linear functional over that shape is 0, and this definition requires only vector space properties. Similarly, the quasi-center definition requires only vector space properties, such as the ability to divide any line segment in a given ratio. Both the quasi-center and the centroid are therefore preserved under linear transformations. Suppose triangle ABC in Figure 8 were coordinatized so that A was at the origin of the vector space $\mathbb{R}^2$. Then a suitable linear transformation could move the vectors AB and AC to any two (linearly independent) vectors, thus defining any desired triangle. For any of these triangles, the set $P_{1/3}$ would be constructed by sliding a $1/3$ homothety of the triangle around its perimeter, and this set would always consist of a single point, which is simultaneously the quasi-center and the centroid. The vector space axioms are sufficient to define both the homothety and the sliding, so no metric notions are needed.

Since the centroid and quasi-center agree on all triangles, a natural conjecture is that the quasi-center is just an elaborate redefinition of the centroid. The quarter-circle in Figure 9 shows that this conjecture is not true. We can calculate the quasi-center q for the quarter-circle of radius 1. By symmetry, the quasi-center must be on the line AD. Some homothetic copies of the quarter-circle are shown, of factor a. They are a times the quarter-circle of radius 1, so their radius is a. The homothetic copies are slid along the quarter-circle perimeter. If a is chosen correctly, then they will sweep over all the interior except the single point q. The figure shows that the length of AD is $(1+\sqrt{2})a$. Since AD is also a radius of the

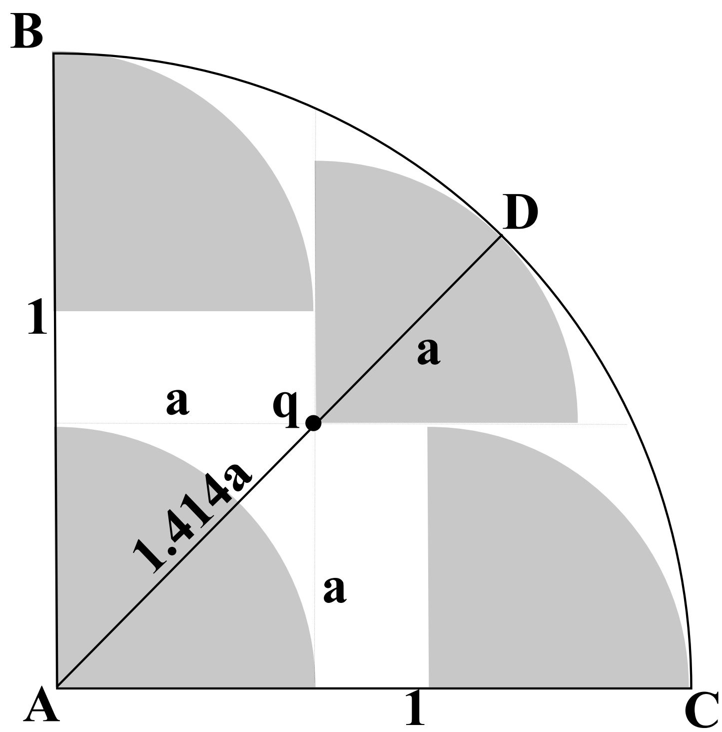

Figure 9: A Convex Shape for which the Quasi-Center and Centroid Differ

quarter-circle, its length is 1, so

$$(1+\sqrt{2})a = 1 \quad (7)$$
$$a \simeq 0.4142. \quad (8)$$

If a Cartesian coordinate system were applied to the quarter-circle, then the quasi-center would occur at $(0.4142, 0.4142)$. The centroid of the quarter-circle, however, is given by $(4/(3\pi), 4/(3\pi))$, which is approximately equal to $(0.4244, 0.4244)$. This example suffices to show that the centroid and the quasi-center are not always the same point. Their difference, though, in this case and in a few others that were looked at, is often not very great, and might not be of much practical importance. Perhaps this result is not surprising, since both definitions formalize the intuitive notion of a "center."

While the quasi-center has an appealing intuitive construction, the centroid has some advantages that the quasi-center lacks. First, the centroid has a natural physical interpretation: the centroid is the point at which a planar convex set (of uniform density) will balance. In three dimensions, the same balancing quality appears, for example in mechanics, where the motion of an irregular object can often be described solely with reference to its centroid. The quasi-center does not (at least not yet) have any physical interpretation beyond its construction. Second, the centroid definition applies equally well to non-convex sets. As long as a set satisfies some very mild measurability conditions, and is bounded, its centroid can be calculated; even some unbounded sets have centroids. The quasi-center, on the other hand, relies heavily on convexity, using chords that span the entire set. Third, the centroid is a unique point for a convex set of arbitrary dimension. While the quasi-center is unique for many higher-dimensional sets, uniqueness is only guaranteed for planar sets, and a triangular prism provides a simple instance of non-uniqueness in three dimensions. In general, then, the quasi-center is of more theoretical than practical interest, in comparison to the centroid.

5 Summary

This paper has introduced the quasi-center of a compact convex planar set. The quasi-center can be constructed by a geometrically intuitive sliding homothety construction. For planar sets, the quasi-center exists and is unique. In higher dimensions, at least one quasi-center exists, but it is not necessarily unique. Intuitively, the quasi-center is centrally located within a convex planar set, and generalizes the technical definition of a convex set's *center*, which is also an intuitive definition, but only exists for very symmetrical sets. The quasi-center agrees with the centroid for many planar sets, such as triangles, but the example of a quarter-circle shows that the the quasi-center and the centroid are not necessarily identical—although, in many examples, they are very close. In comparison to the centroid, which exists and is unique for a wide variety of sets, both convex and non-convex, in arbitrary dimensions, the quasi-center is more of a theoretical curiosity than a practical tool.

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